Gauge Anomaly associated with the Majorana Fermion in $8k + 1$ dimensions

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Abstract

Using an elementary method, we show that an odd number of Majorana fermions in $8k+1$ dimensions suffer from a gauge anomaly that is analogous to the Witten global gauge anomaly. This anomaly cannot be removed without sacrificing the perturbative gauge invariance. Our construction of higher-dimensional examples ($k \geq 1$) makes use of the SO(8) instanton on S^8.

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§1. Introduction

The Majorana fermion in $8k+1$-dimensional Minkowski spacetime is very peculiar. The charge conjugation matrix is symmetric $C^T = +C$ in $8k+1$ dimensions and only possible Lorentz invariant mass term for a single Majorana fermion, $m\psi^T C^{-1}\psi$, identically vanishes. This implies that one cannot apply the gauge invariant Pauli-Villars regularization to a single Majorana fermion. This situation generally persists for odd number of Majorana fermions.

A similar phenomenon is also found with the lattice regularization.[1] The most impressive example is given by the 1-dimensional ($k=0$) case, i.e., quantum mechanics, for which a lattice action of a free Majorana fermion $\widetilde{\psi}$ in the momentum space would be

$$\int_{-\pi/a}^{\pi/a} dp\, \widetilde{\psi}^T(-p)\widetilde{D}(p)\widetilde{\psi}(p), \tag{1·1}$$

where a denotes the lattice spacing. The variable $\widetilde{\psi}$ has a gauge index but no spinor index. Since $\widetilde{\psi}$ is Grassmann-odd, the kernel $\widetilde{D}(p)$ must be odd under $p \leftrightarrow -p$. Also, $\widetilde{D}(p)$ must be periodic in the Brillouin zone $\widetilde{D}(p+2\pi/a) = \widetilde{D}(p)$ for locality. These two requirements, however, imply that the kernel inevitably possesses a zero at $p = \pi/a$ which corresponds to the species doubler. Thus one cannot write down a lattice action of a free single Majorana fermion which is consistent with locality, at least in the form of the ansatz (1·1).

These "phenomenological" observations strongly suggest the existence of some sort of anomaly associated to the Majorana fermion in $8k+1$ dimensions. Since there is no perturbative (or local) gauge anomaly in these *odd* dimensions, one expects that something similar to the global gauge anomaly in even-dimensional spaces[2,3] occurs. This possibility has been suggested in ref. 4) and the global *gravitational* anomaly[5] for the Majorana fermion in these dimensions has been demonstrated. Also the global *gauge* anomaly associated to the Majorana fermion in 1 dimension ($k=0$) has been known.[6] For example, supersymmetric quantum mechanics which describes a spinning particle in curved spacetime suffers from the global "gauge" anomaly, when the spacetime does not admit a spin structure. To our knowledge, however, any examples of the global gauge anomaly associated to the Majorana fermion in higher ($k \geq 1$) dimensions have not been given.

In this paper, we provide an example of the gauge anomaly for the Majorana fermion in $8k+1$ dimensions in the $(\mathbf{N}, \mathbf{8_v}, \ldots, \mathbf{8_v})$ representation of the gauge group $SO(N) \times SO(8)^k$ ($N \geq 3$). After starting with generalities concerning with the problem, we describe our strategy to demonstrate the existence of gauge anomaly in that system with base space of topology $S^1 \times S^8 \times \cdots \times S^8$.

The Euclidean action for the Majorana fermion is given by

$$S = \int \mathrm{d}^{8k+1} x \, \psi^T(x) \slashed{D}_{8k+1} \psi(x), \tag{1.2}$$

where the covariant derivative is defined by $\slashed{D}_{8k+1} = \sum_{\mu=1}^{8k+1} \gamma_\mu \{\partial_\mu + A_\mu(x)\}$ and we have chosen a representation in which all γ_μ are real and symmetric. The gauge potential $A_\mu(x)$ takes the value in the fundamental representation of the Lie algebra of $\mathrm{SO}(N) \times \mathrm{SO}(8)^k$ and $A_\mu^* = A_\mu = -A_\mu$. The Dirac operator $\slashed{D}_{8k+1}$ is thus real and anti-symmetric. The partition function is formally given by the Pfaffian of the Dirac operator,

$$\mathrm{Pf}\{i\slashed{D}_{8k+1}\}, \tag{1.3}$$

whose square is the Dirac determinant

$$\det\{i\slashed{D}_{8k+1}\} = (\mathrm{Pf}\{i\slashed{D}_{8k+1}\})^2. \tag{1.4}$$

The Dirac determinant can be defined in a gauge-invariant manner by taking an appropriately regularized product of all eigenvalues λ_n of $i\slashed{D}_{8k+1}$:

$$i\slashed{D}_{8k+1}\varphi_n(x) = \lambda_n \varphi_n(x). \tag{1.5}$$

We remark that all the eigenvalues λ_n are real and, because φ_n^* is an eigenfunction with the eigenvalue $-\lambda_n$, nonzero eigenvalues come in pairs $(\lambda_n, -\lambda_n)$. Equation (1.4) indicates that the Majorana Pfaffian is defined by the product of either the eigenvalue λ_n or the eigenvalue $-\lambda_n$ for each n. The choice of λ_n or $-\lambda_n$, however, leads to an ambiguity in the sign of the product. We would like to remove this ambiguity so that the Majorana Pfaffian is a smooth function of gauge-field configurations to ensure that the Schwinger-Dyson equations hold.[2] We can define $\mathrm{Pf}\{i\slashed{D}_{8k+1}\}[A]$ for a particular gauge field A as the product of (say) the positive λ_n. Then, by requiring the smoothness of the Pfaffian, there is no sign ambiguity for other gauge-field configurations in each topological sector.

Our prime interest regards the possibility that the Majorana Pfaffian (1.3) may not be invariant under a certain gauge transformation, defined as

$$A_\mu^g(x) \equiv g(x)^{-1}\{\partial_\mu + A_\mu(x)\}g(x). \tag{1.6}$$

Because A and A^g give an identical set of eigenvalues, the Pfaffian is gauge invariant up to a sign:

$$\mathrm{Pf}\{i\slashed{D}_{8k+1}\}[A^g] = \pm\,\mathrm{Pf}\{i\slashed{D}_{8k+1}\}[A]. \tag{1.7}$$

By noting the smoothness of the Pfaffian, one can show that the sign in Eq. (1.7) does not change under any smooth deformation of the gauge field A; the sign depends only on g and,

possibly, on the topology of A. We may, therefore, adopt a convenient form of the gauge field A when computing the sign in Eq. (1·7).

In § 2, we first show the existence of a gauge anomaly in as $SO(N)$ gauge system with an odd number of Majorana fermions on $\mathbb{R}^1$ or S^1 ($k = 0$). In § 3, we find the gauge field A and gauge transformation g which lead to a minus sign in Eq. (1·7) through the mechanism in the 1-dimensional system presented in § 2, supplemented with the appropriate number of $SO(8)$ instantons.[7] Through its construction, the base manifold is found to take the form of the topology of $S^1 \times S^8 \times \cdots \times S^8$. A conclusion is given in § 4.

§2.   One-dimensional case

For $k = 0$, the gamma matrix is a scalar, specifically, $\gamma_1 = 1$, and the Dirac operator can thus be written as $D_1 = \partial_1 + A_1$. We take the following form of the gauge potential:

$$A_1(x_1) = A_1^1(x_1)T^1, \quad T^1 = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}. \tag{2·1}$$

Here T^1 is one of the generators of $SO(N)$. For this special configuration, we can explicitly solve the eigenvalue problem (1·5). [See Exercise 5.8 of Ref. 8) for a similar problem in a U(1) theory.] We impose the periodic boundary condition $\varphi_n(+L/2) = \varphi_n(-L/2)$ on a finite interval $-L/2 \le x_1 \le +L/2$. The eigenvalues are then given by

$$\lambda_n L = \pm a + 2\pi n, \quad n \in \mathbb{Z}, \tag{2·2}$$

where

$$a = \int_{-L/2}^{+L/2} dx_1 \, A_1^1(x) = -\frac{1}{2} \int_{-L/2}^{+L/2} dx_1 \, \mathrm{tr}\{T^1 A_1(x_1)\}, \tag{2·3}$$

and

$$\lambda_n L = 2\pi n, \quad n \in \mathbb{Z}. \tag{2·4}$$

The latter eigenvalues are $N - 2$-fold degenerate.

The Dirac determinant is given by the product of all the eigenvalues in Eqs. (2·2) and (2·4). This product can be regularized by dividing it by the determinant of the free Dirac operator:

$$\frac{\det'\{iD_1\}[A]}{\det'\{iD_1\}[0]} = -\frac{a^2}{L^2} \prod_{n=1}^{\infty} \left(1 - \frac{a^2}{4\pi^2 n^2}\right)^2 = -\frac{4}{L^2} \sin^2\left(\frac{a}{2}\right), \tag{2·5}$$

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where the prime indicates that zero eigenvalues independent of the gauge field are omitted from the product. From this, we have

$$\frac{\text{Pf}'\{iD_1\}[A]}{\text{Pf}'\{iD_1\}[0]} = \frac{2i}{L} \sin\left(\frac{a}{2}\right), \qquad (2\cdot6)$$

because this defines a smooth function of the gauge field A. If the Pfaffian were defined by the absolute value $(2i/L)|\sin(a/2)|$ instead, it would develop a cusp at $a = 0$, and, consequently, the Schwinger-Dyson equations would not hold.

Now, we consider a particular gauge transformation,

$$g(x_1) = e^{\theta^1(x_1)T^1}. \qquad (2\cdot7)$$

To conclude the existence of a global anomaly, it is sufficient to show that the sign in Eq. (1·7) is minus for the gauge transformation which satisfies the boundary condition $g(-L/2) = g(+L/2) = 1$. The real function $\theta^1(x_1)$ in Eq. (2·7) thus satisfies

$$\theta^1(-L/2) = 2\pi w_-, \quad \theta^1(+L/2) = 2\pi w_+, \quad w_\pm \in \mathbb{Z}. \qquad (2\cdot8)$$

From Eqs. (1·6), (2·7) and (2·8), it is found that the integral (2·3) changes under the gauge transformation as

$$a^g = -\frac{1}{2}\int_{-L/2}^{+L/2} \mathrm{d}x_1 \ \text{tr}\{T^1 A_1^g(x_1)\} = a + 2\pi w, \qquad (2\cdot9)$$

where $w \equiv w_+ - w_-$. From Eq. (2·6), we see that the Pfaffian actually changes sign under the gauge transformation

$$\text{Pf}\{iD_1\}[A^g] = (-1)^w \, \text{Pf}\{iD_1\}[A] \qquad (2\cdot10)$$

when the "winding number" w of the gauge transformation is odd. As we noted in the introduction, this relation should hold for all gauge-field configurations A that can be smoothly deformed into the above particular configuration (2·1).

It is legitimate to regard $(-1)^w$ in Eq. (2·10) as a gauge anomaly, because it cannot be removed by any local counterterm. More precisely, it can be removed *only by sacrificing the local (or perturbative) gauge invariance*. In fact, the anomaly $(-1)^w$ is removed by adding the local term

$$\Delta S = \frac{i}{4}\int_{-L/2}^{+L/2} \mathrm{d}x_1 \ \text{tr}\{T^1 A_1(x_1)\} \qquad (2\cdot11)$$

to the action, because this term changes as $\Delta S \to \Delta S - i\pi w$ under the gauge transformation (2·7) [see Eq. (2·9)]. This term, however, manifestly breaks the local SO(N) gauge

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symmetry for $N \geq 3$. In this respect, the present gauge anomaly should be clearly distinguished from the apparent global gauge anomaly associated with the Dirac fermion in an odd number of dimensions,[9] which can always be removed by the Chern-Simons term. We conclude that a single Majorana fermion in 1 dimension suffers from the global gauge anomaly and, in general, cannot be defined in a gauge invariant way. This conclusion also holds in the case that the number of Majorana fermions is odd.

In order to find a higher-dimensional analogue of the above-described phenomenon, it is useful to observe the anomaly from the point of view of a flow of eigenvalues of iD_1. For this purpose, introduce an additional coordinate x_0 ($-\infty < x_0 < +\infty$), and the one-parameter family of gauge fields, $A_1(x_0, x_1)$, which adiabatically interpolates between our previous gauge field (2·1) in the limit $x_0 \to -\infty$ and the gauge transformed one A^g in the limit $x_0 \to +\infty$. The Pfaffian for $x_0 \to -\infty$, $\text{Pf}\{iD_1\}[A]$, is defined by the product of (say) positive eigenvalues of iD_1. The Pfaffian for $x_0 \to +\infty$, $\text{Pf}\{iD_1\}[A^g]$, is then uniquely given by the product of the eigenvalues of the eigenfunctions adopted for $x_0 \to -\infty$. The global gauge anomaly occurs as a result of a flow of eigenvalues along x_0 in which an odd number of positive eigenvalues of iD_1 for $x_0 \to -\infty$ becomes negative for $x_0 \to +\infty$. (See Ref. 2).) Such a flow of eigenvalues is closely related to the number of right-handed zero modes of the Dirac operator in 2 dimensions,

$$\slashed{D}_2 = \sigma^2 \frac{\partial}{\partial x_0} + \sigma^1 D_1, \tag{2·12}$$

where σ^1 and σ^2 denote the Pauli matrices, and the gauge potential in D_1 is $A_1(x_0, x_1)$. A zero mode Ψ, for which $\slashed{D}_2 \Psi = 0$, with right-handed chirality, expressed as $\sigma^3 \Psi = +\Psi$, satisfies $\frac{\partial \Psi}{\partial x_0} = iD_1 \Psi$. In the adiabatic approximation, this is solved as

$$\Psi(x_0, x_1) = \begin{pmatrix} \exp\{+ \int_0^{x_0} dx_0' \, \lambda_n(x_0')\} \varphi_n(x_0, x_1) \\ 0 \end{pmatrix}. \tag{2·13}$$

This wave function is normalizable iff $\lambda_n(x_0)$ is positive for $x_0 \to -\infty$ and negative for $x_0 \to +\infty$. Hence, a flow of eigenvalues exhibits the global gauge anomaly iff the number of right-handed zero modes of $\slashed{D}_2$ is odd. The above study of the 1-dimensional case shows that the number of right-handed zero modes of $\slashed{D}_2$ is odd for g in Eq. (2·7) with an odd winding number w.

Similarly, left-handed zero modes of $\slashed{D}_2$ correspond to eigenfunctions for which $\lambda_n(x_0)$ is negative for $x_0 \to -\infty$ and positive for $x_0 \to +\infty$. As noted above, eigenvalues of D_1 come in pairs, $\pm \lambda_n$. Thus the number of right-handed zero modes of $\slashed{D}_2$ is the same as the number of left-handed ones. In other words, the index of $\slashed{D}_2$ identically vanishes: $\text{index}\{i\slashed{D}_2\} = 0$.

§3. $8k + 1$-dimensional case

Let us now turn to the $8k + 1$-dimensional problem. For this, we introduce the Dirac operator in $8k + 2$ dimensions

$$\displaystyle{\not{D}_{8k+2} = \not{D}_2 \otimes 1 + \sigma^3 \otimes \not{D}_{8k},}$$

$$\not{D}_{8k} = \sum_{\mu=2}^{8k+1} \gamma_\mu(\partial_\mu + A_\mu). \tag{3.1}$$

We choose such a representation that all $8k$-dimensional gamma matrices γ_μ ($\mu = 2, 3, \ldots, 8k + 1$) are real. The corresponding chiral matrices are given by

$$\Gamma^{(8k+2)} = \sigma^3 \otimes \gamma^{(8k)}, \quad \gamma^{(8k)} = \gamma_2 \gamma_3 \cdots \gamma_{8k+1}. \tag{3.2}$$

To show that the minus sign is realized in Eq. (1.7), it is enough to find (A_μ, g) such that the number of right-handed zero modes of $\not{D}_{8k+2}$ is odd, as in the $k = 0$ case in the previous section.[*] For this purpose, we choose $A_1(x_0, x_1)$ to be identical to the above 1-dimensional example with w odd. It thus takes values in the Lie algebra of $SO(N)$. Other components of the gauge field, A_μ ($\mu = 2, 3, \ldots, 8k + 1$), are assumed to take values in the Lie algebra of $SO(8)^k$ and depend only on $x_2, x_3, \ldots, x_{8k+1}$. Under these assumptions, we have

$$\not{D}_{8k+2}^\dagger \not{D}_{8k+2} = \not{D}_2^\dagger \not{D}_2 \otimes 1 + 1 \otimes \not{D}_{8k}^\dagger \not{D}_{8k}, \tag{3.3}$$

and the equation $\not{D}_{8k+2}\Psi = 0$ can be solved through the "separation of variables" $\Psi = \xi(x_0, x_1) \otimes \phi(x_2, \ldots, x_{8k+1})$, where $\not{D}_2\xi = 0$ and $\not{D}_{8k}\phi = 0$. From Eq. (3.2), a right-handed zero mode $\Gamma^{(8k+2)}\Psi = +\Psi$ is obtained if (i) $\sigma^3\xi = +\xi$ and $\gamma^{(8k)}\phi = +\phi$, or (ii) $\sigma^3\xi = -\xi$ and $\gamma^{(8k)}\phi = -\phi$. Let us denote the number of ξ satisfying $\sigma^3\xi = +\xi$ by n. As noted above, n is odd and the same as the number of ξ satisfying $\sigma^3\xi = -\xi$. Let us further denote the number of ϕ satisfying $\gamma^{(8k)}\phi = \pm\phi$ by $N_\pm$; i.e., $N_+ - N_- = \text{index}\{i\not{D}_{8k}\}$. From these facts, we see that the number of right-handed zero modes of $\not{D}_{8k+2}$ is given by

$$n(N_+ + N_-) = n(2N_- + \text{index}\{i\not{D}_{8k}\}). \tag{3.4}$$

Since n is odd, the number of right-handed zero modes of $\not{D}_{8k+2}$ is odd if $\text{index}\{i\not{D}_{8k}\}$ is odd.

We are now led to ask if there is a gauge-field configuration which provides an odd index for $\not{D}_{8k}$. In fact there is, and such a configuration can be constructed by using the so-called

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[*] It can be confirmed that $\not{D}_{8k+2}\Psi = 0$ with $\Gamma^{(8k+2)}\Psi = +\Psi$ is equivalent to the non-trivial flow of an eigenvalue of $i\not{D}_{8k+1}$, in which the last gamma matrix is given by $\gamma^{(8k)}$. It can also easily be shown that the number of zero modes of $\not{D}_{8k+2}$ modulo 4 is invariant under a smooth deformation of the gauge field and is either 0 or 2 (mod 4); this thus defines an index. What we want to find is an appropriate case for which this index is 2 (mod 4).

SO(8) instanton in 8 dimensions.[7] The gauge potential 1-form of the SO(8) instanton on S^8 is given, after the stereographic projection to $\mathbb{R}^8$, by

$$A(y) = \frac{y^2}{y^2 + \rho^2} h^{-1}(y)\mathrm{d}h(y), \quad y^2 \equiv \sum_{\alpha=1}^{8} y_\alpha y_\alpha, \tag{3.5}$$

where y_α denote the orthogonal coordinate system of $\mathbb{R}^8$ and ρ is the size of instanton. The instanton is asymptotically pure-gauge with the gauge transformation

$$h(y) = \frac{i\sum_{m=1}^{6} y_m\sigma_m + iy_7\sigma + y_8}{(y^2)^{1/2}} \in \mathrm{SO}(8), \tag{3.6}$$

where σ_m represents the gamma matrices in 6 dimensions, which satisfy $\{\sigma_m, \sigma_n\} = 2\delta_{mn}$ (m and n take the values 1, 2, …, 6) and $\sigma = -i\sigma_1\sigma_2\cdots\sigma_6$. The gamma matrices are 8×8 and taken to be purely imaginary. From the gauge potential, we have the field strength 2-form $F = \mathrm{d}A + A \wedge A$,

$$F(y) = \frac{2\rho^2}{(y^2 + \rho^2)^2} \sum_{\alpha,\beta=1}^{8} \sigma_{\alpha\beta}\,\mathrm{d}y_\alpha \wedge \mathrm{d}y_\beta, \tag{3.7}$$

where the 8×8 real and anti-symmetric matrices $\sigma_{\alpha\beta}$ are

$$\sigma_{mn} = \frac{1}{4}[\sigma_m, \sigma_n], \quad \sigma_{m7} = \frac{1}{2}\sigma_m\sigma = -\sigma_{7m},$$
$$\sigma_{m8} = -\frac{i}{2}\sigma_m = -\sigma_{8m}, \quad \sigma_{78} = -\frac{i}{2}\sigma = -\sigma_{87}. \tag{3.8}$$

They form the SO(8) Lie algebra in the $\mathbf{8_s}$ representation. The field strength is self-dual in the sense that $*(F \wedge F) = +F \wedge F$ and the instanton has a unit 4th Chern number

$$\frac{1}{(2\pi)^4 4!} \int_{S^8} \mathrm{tr}\{F^4\} = +1. \tag{3.9}$$

Using this SO(8) instanton on S^8, we construct the gauge-field configuration in $8k + 2$ dimensions as follows. We take our base manifold as $S^1 \times S^1 \times \prod_{i=1}^{k}\{S^8\}$ and denote the coordinates of the first $S^1 \times S^1$ by (x_0, x_1) and the (stereographically projected) coordinates of the ith S^8 collectively by $y^{(i)}$. On the ith S^8, we place the SO(8) instantons $A(y^{(i)})$ that take values in the ith factor of the gauge group SO(8)^k. From the standard index theorem,

$$\mathrm{index}\{i\slashed{D}_{8k}\} = \frac{1}{(2\pi)^{4k}(4k)!} \int \mathrm{tr}\{F^{4k}\}, \tag{3.10}$$

which is valid for our base manifold $\prod_{i=1}^{k}\{S^8\}$, and Eq. (3.9), we see that that the above configuration gives $\mathrm{index}\{i\slashed{D}_{8k}\} = +1$. Finally, from Eq. (3.4), we see that the number of right-handed zero modes of $\slashed{D}_{8k+2}$ is odd with the above gauge-field configuration.

Returning to our original problem, the above configuration for $x_0 \to +\infty$ corresponds to A^g in Eq. (1·7) and can be written as $g(x_1)^{-1}(\mathrm{d} + \widetilde{A})g(x_1)$ with a certain gauge potential $\widetilde{A}$. Combining all the above arguments, we infer that the Majorana Pfaffian (1·3) changes sign under the gauge transformation (2·7) in the presence of k SO(8) instantons along $8k$ dimensions. This phenomenon can properly be regarded as the gauge anomaly, because, as in the $k = 0$ case, this anomaly can be removed only by sacrificing the perturbative gauge invariance.

In two aspects, the gauge anomaly observed above in the case of higher dimensions is different from the Witten global gauge anomaly. First, we have shown the non-invariance $\mathrm{Pf}\{i\slashed{D}_{8k+1}\}[A^g] = -\mathrm{Pf}\{i\slashed{D}_{8k+1}\}[A]$ for the gauge transformation $g(x_1)$ in Eq. (2·7). However, the gauge transformation $g(x_1)$ does not approach the identity even at the infinity of $\mathbb{R}^{8k}$ (in the sense of the stereographic projection), and the transformation is not localized in all directions. In conventional treatments of the global gauge anomaly, gauge transformations are restricted to a class of transformations which approach the identity at the spacetime infinity. Such gauge transformations in d-dimensional Euclidean space are classified by $\pi_d(G)$, where G is the gauge group. Second, the gauge field with which we have shown the non-invariance requires the presence of instantons that cannot be smoothly deformed to $A = 0$. This situation is also different from that for the Witten global gauge anomaly, which exists even for $A = 0$. The SO(8) instanton given in Eq. (3·5) neither is a solution of the Euclidean Yang-Mills theory $\int \mathrm{tr}\{F \wedge *F\}$ nor has a finite action in $\mathbb{R}^8$. There is a possibility, however, that such a configuration becomes relevant in a theory with higher-dimensional operators in the action, possibly in the context of string theory. (See, for example, Ref. 10).)

§4. Conclusion

In this paper, we have demonstrated a sort of global gauge anomaly associated with the Majorana fermion in $8k + 1$ dimensions. Possible implications of this phenomenon should be investigated. We expect interesting applications in quantum mechanics ($k = 0$) and in string theory ($k = 1$).

Acknowledgements

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